But $\sqrt{(\Delta x)^2 + (\Delta y)^2} = |\Delta z|$, and so

$$\left|\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta z}\right| = 1.$$

Also, $\varepsilon_1 + i\varepsilon_2$ tends to 0 as $(\Delta x, \Delta y)$ approaches (0, 0). So the last term on the right in equation (5) tends to 0 as the variable $\Delta z = \Delta x + i\Delta y$ tends to 0. This means that the limit of the left-hand side of equation (5) exists and that

$$f'(z_0) = u_x + iv_x, \tag{6}$$

where the right-hand side is to be evaluated at (x_0, y_0).

EXAMPLE 1. Consider the exponential function

$$f(z) = e^z = e^x e^{iy} \qquad (z = x + iy),$$

some of whose mapping properties were discussed in Sec. 13. In view of Euler's formula (Sec. 6), this function can, of course, be written

$$f(z) = e^x \cos y + ie^x \sin y,$$

where y is to be taken in radians when $\cos y$ and $\sin y$ are evaluated. Then

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

Since $u_x = v_y$ and $u_y = -v_x$ everywhere and since these derivatives are everywhere continuous, the conditions in the theorem are satisfied at all points in the complex plane. Thus $f'(z)$ exists everywhere, and

$$f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y.$$

Note that $f'(z) = f(z)$.

EXAMPLE 2. It also follows from the theorem in this section that the function $f(z) = |z|^2$, whose components are

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0,$$

has a derivative at $z = 0$. In fact, $f'(0) = 0 + i0 = 0$ (compare Example 2, Sec. 18). We saw in Example 2, Sec. 20, that this function *cannot* have a derivative at any nonzero point since the Cauchy–Riemann equations are not satisfied at such points.

22. POLAR COORDINATES

Assuming that $z_0 \neq 0$, we shall in this section use the coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1}$$

to restate the theorem in Sec. 21 in polar coordinates.

Depending on whether we write

$$z = x + iy \quad \text{or} \quad z = re^{i\theta} \qquad (z \neq 0)$$

when $w = f(z)$, the real and imaginary parts of $w = u + iv$ are expressed in terms of either the variables x and y or r and θ. Suppose that the first-order partial derivatives of u and v with respect to x and y exist everywhere in some neighborhood of a given nonzero point z_0 and are continuous at that point. The first-order partial derivatives with respect to r and θ also have these properties, and the chain rule for differentiating real-valued functions of two real variables can be used to write them in terms of the ones with respect to x and y. More precisely, since

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta},$$

one can write

$$u_r = u_x \cos\theta + u_y \sin\theta, \qquad u_\theta = -u_x r \sin\theta + u_y r \cos\theta. \tag{2}$$

Likewise,

$$v_r = v_x \cos\theta + v_y \sin\theta, \qquad v_\theta = -v_x r \sin\theta + v_y r \cos\theta. \tag{3}$$

If the partial derivatives with respect to x and y also satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{4}$$

at z_0, equations (3) become

$$v_r = -u_y \cos\theta + u_x \sin\theta, \qquad v_\theta = u_y r \sin\theta + u_x r \cos\theta \tag{5}$$

at that point. It is then clear from equations (2) and (5) that

$$ru_r = v_\theta, \quad u_\theta = -rv_r \tag{6}$$

at the point z_0.

If, on the other hand, equations (6) are known to hold at z_0, it is straightforward to show (Exercise 7) that equations (4) must hold there. Equations (6) are, therefore, an alternative form of the Cauchy–Riemann equations (4).

We can now restate the theorem in Sec. 21 using polar coordinates.

Theorem. *Let the function*

$$f(z) = u(r, \theta) + iv(r, \theta)$$

be defined throughout some ε neighborhood of a nonzero point $z_0 = r_0 \exp(i\theta_0)$, and suppose that the first-order partial derivatives of the functions u and v with respect to r

and θ exist everywhere in that neighborhood. If those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar form

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

of the Cauchy–Riemann equations at (r_0, θ_0), then $f'(z_0)$ exists.

The derivative $f'(z_0)$ here can be written (see Exercise 8)

$$f'(z_0) = e^{-i\theta}(u_r + iv_r), \tag{7}$$

where the right-hand side is to be evaluated at (r_0, θ_0).

EXAMPLE 1. Consider the function

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta) \qquad (z \neq 0). \tag{8}$$

Since

$$u(r,\theta) = \frac{\cos\theta}{r} \quad \text{and} \quad v(r,\theta) = -\frac{\sin\theta}{r},$$

the conditions in the above theorem are satisfied at every nonzero point $z = re^{i\theta}$ in the plane. In particular, the Cauchy–Riemann equations

$$ru_r = -\frac{\cos\theta}{r} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sin\theta}{r} = -rv_r$$

are satisfied. Hence the derivative of f exists when $z \neq 0$; and, according to expression (7),

$$f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right) = -e^{-i\theta}\frac{e^{-i\theta}}{r^2} = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}.$$

EXAMPLE 2. The theorem can be used to show that, when α is a fixed real number, the function

$$f(z) = \sqrt[3]{r}e^{i\theta/3} \qquad (r > 0, \alpha < \theta < \alpha + 2\pi) \tag{9}$$

has a derivative everywhere in its domain of definition. Here

$$u(r,\theta) = \sqrt[3]{r}\cos\frac{\theta}{3} \quad \text{and} \quad v(r,\theta) = \sqrt[3]{r}\sin\frac{\theta}{3}.$$

Inasmuch as

$$ru_r = \frac{\sqrt[3]{r}}{3}\cos\frac{\theta}{3} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sqrt[3]{r}}{3}\sin\frac{\theta}{3} = -rv_r$$

and since the other conditions in the theorem are satisfied, the derivative $f'(z)$ exists at each point where $f(z)$ is defined. Furthermore, expression (7) tells us that

$$f'(z) = e^{-i\theta}\left[\frac{1}{3\left(\sqrt[3]{r}\right)^2}\cos\frac{\theta}{3} + i\frac{1}{3\left(\sqrt[3]{r}\right)^2}\sin\frac{\theta}{3}\right],$$

or

$$f'(z) = \frac{e^{-i\theta}}{3\left(\sqrt[3]{r}\right)^2}e^{i\theta/3} = \frac{1}{3\left(\sqrt[3]{r}e^{i\theta/3}\right)^2} = \frac{1}{3\left[f(z)\right]^2}.$$

Note that when a specific point z is taken in the domain of definition of f, the value $f(z)$ is one value of $z^{1/3}$ (see Sec. 11). Hence this last expression for $f'(z)$ can be put in the form

$$\frac{d}{dz}z^{1/3} = \frac{1}{3\left(z^{1/3}\right)^2}$$

when that value is taken. Derivatives of such power functions will be elaborated on in Chap. 3 (Sec. 32).

EXERCISES

1. Use the theorem in Sec. 20 to show that $f'(z)$ does not exist at any point if

(*a*) $f(z) = \bar{z}$; (*b*) $f(z) = z - \bar{z}$; (*c*) $f(z) = 2x + ixy^2$; (*d*) $f(z) = e^x e^{-iy}$.

2. Use the theorem in Sec. 21 to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when

(*a*) $f(z) = iz + 2$; (*b*) $f(z) = e^{-x}e^{-iy}$;

(*c*) $f(z) = z^3$; (*d*) $f(z) = \cos x \cosh y - i \sin x \sinh y$.

Ans. (*b*) $f''(z) = f(z)$; (*d*) $f''(z) = -f(z)$.

3. From results obtained in Secs. 20 and 21, determine where $f'(z)$ exists and find its value when

(*a*) $f(z) = 1/z$; (*b*) $f(z) = x^2 + iy^2$; (*c*) $f(z) = z \operatorname{Im} z$.

Ans. (*a*) $f'(z) = -1/z^2$ $(z \neq 0)$; (*b*) $f'(x + ix) = 2x$; (*c*) $f'(0) = 0$.

4. Use the theorem in Sec. 22 to show that each of these functions is differentiable in the indicated domain of definition, and then use expression (7) in that section to find $f'(z)$:

(*a*) $f(z) = 1/z^4$ $(z \neq 0)$;

(*b*) $f(z) = \sqrt{r}e^{i\theta/2}$ $(r > 0, \alpha < \theta < \alpha + 2\pi)$;

(*c*) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$ $(r > 0, 0 < \theta < 2\pi)$.

Ans. (*b*) $f'(z) = \dfrac{1}{2f(z)}$; (*c*) $f'(z) = i\dfrac{f(z)}{z}$.

5. Show that when $f(z) = x^3 + i(1-y)^3$, it is legitimate to write

$$f'(z) = u_x + iv_x = 3x^2$$

only when $z = i$.

6. Let u and v denote the real and imaginary components of the function f defined by the equations

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z} & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Verify that the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at the origin $z = (0, 0)$. [Compare Exercise 9, Sec. 19, where it is shown that $f'(0)$ nevertheless fails to exist.]

7. Solve equations (2), Sec. 22, for u_x and u_y to show that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, \quad u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

Then use these equations and similar ones for v_x and v_y to show that, in Sec. 22, equations (4) are satisfied at a point z_0 if equations (6) are satisfied there. Thus complete the verification that equations (6), Sec. 22, are the Cauchy–Riemann equations in polar form.

8. Let a function $f(z) = u + iv$ be differentiable at a nonzero point $z_0 = r_0 \exp(i\theta_0)$. Use the expressions for u_x and v_x found in Exercise 7, together with the polar form (6), Sec. 22, of the Cauchy–Riemann equations, to rewrite the expression

$$f'(z_0) = u_x + iv_x$$

in Sec. 21 as

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where u_r and v_r are to be evaluated at (r_0, θ_0).

9. (*a*) With the aid of the polar form (6), Sec. 22, of the Cauchy–Riemann equations, derive the alternative form

$$f'(z_0) = \frac{-i}{z_0}(u_\theta + iv_\theta)$$

of the expression for $f'(z_0)$ found in Exercise 8.

(*b*) Use the expression for $f'(z_0)$ in part (*a*) to show that the derivative of the function $f(z) = 1/z$ $(z \neq 0)$ in Example 1, Sec. 22, is $f'(z) = -1/z^2$.

10. (*a*) Recall (Sec. 5) that if $z = x + iy$, then

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.$$

By *formally* applying the chain rule in calculus to a function $F(x, y)$ of two real variables, derive the expression

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

(*b*) Define the operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

suggested by part (*a*), to show that if the first-order partial derivatives of the real and imaginary parts of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy–Riemann equations, then

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}[(u_x - v_y) + i(v_x + u_y)] = 0.$$

Thus derive the *complex form* $\partial f/\partial \bar{z} = 0$ *of the Cauchy–Riemann equations.*

23. ANALYTIC FUNCTIONS

We are now ready to introduce the concept of an analytic function. A function f of the complex variable z is *analytic* in an open set if it has a derivative at each point in that set.* If we should speak of a function f that is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S. In particular, f is *analytic at a point* z_0 if it is analytic throughout some neighborhood of z_0.

We note, for instance, that the function $f(z) = 1/z$ is analytic at each nonzero point in the finite plane. But the function $f(z) = |z|^2$ is not analytic at any point since its derivative exists only at $z = 0$ and not throughout any neighborhood. (See Example 2, Sec. 18.)

An *entire* function is a function that is analytic at each point in the entire finite plane. Since the derivative of a polynomial exists everywhere, it follows that *every polynomial is an entire function.*

If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a *singular point,* or singularity, of f. The point $z = 0$ is evidently a singular point of the function $f(z) = 1/z$. The function $f(z) = |z|^2$, on the other hand, has no singular points since it is nowhere analytic.

A necessary, but by no means sufficient, condition for a function f to be analytic in a domain D is clearly the continuity of f throughout D. Satisfaction of the Cauchy–Riemann equations is also necessary, but not sufficient. Sufficient conditions for analyticity in D are provided by the theorems in Secs. 21 and 22.

Other useful sufficient conditions are obtained from the differentiation formulas in Sec. 19. The derivatives of the sum and product of two functions exist wherever the

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*The terms *regular* and *holomorphic* are also used in the literature to denote analyticity.

functions themselves have derivatives. Thus, *if two functions are analytic in a domain D, their sum and their product are both analytic in D.* Similarly, *their quotient is analytic in D provided the function in the denominator does not vanish at any point in D.* In particular, the quotient $P(z)/Q(z)$ of two polynomials is analytic in any domain throughout which $Q(z) \neq 0$.

From the chain rule for the derivative of a composite function, we find that *a composition of two analytic functions is analytic.* More precisely, suppose that a function $f(z)$ is analytic in a domain D and that the image (Sec. 12) of D under the transformation $w = f(z)$ is contained in the domain of definition of a function $g(w)$. Then the composition $g[f(z)]$ is analytic in D, with derivative

$$\frac{d}{dz}g[f(z)] = g'[f(z)]f'(z).$$

The following theorem is especially useful, in addition to being expected.

Theorem. *If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.*

We start the proof by writing $f(z) = u(x, y) + iv(x, y)$. Assuming that $f'(z) = 0$ in D, we note that $u_x + iv_x = 0$; and, in view of the Cauchy–Riemann equations, $v_y - iu_y = 0$. Consequently,

$$u_x = u_y = v_x = v_y = 0$$

at each point in D.

Next, we show that $u(x, y)$ is constant along any line segment L extending from a point P to a point P' and lying entirely in D. We let s denote the distance along L from the point P and let $\mathbf{U}$ denote the unit vector along L in the direction of increasing s (see Fig. 30). We know from calculus that the directional derivative du/ds can be written as the dot product

$$\frac{du}{ds} = (\text{grad } u) \cdot \mathbf{U}, \tag{1}$$

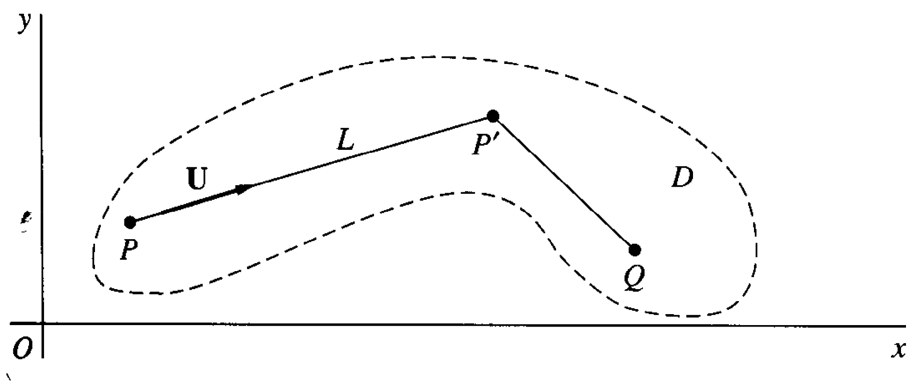

FIGURE 30

where grad u is the gradient vector

$$\text{grad } u = u_x\mathbf{i} + u_y\,\mathbf{j}. \tag{2}$$

Because u_x and u_y are zero everywhere in D, then, grad u is the zero vector at all points on L. Hence it follows from equation (1) that the derivative du/ds is zero along L; and this means that u is constant on L.

Finally, since there is always a finite number of such line segments, joined end to end, connecting any two points P and Q in D (Sec. 10), the values of u at P and Q must be the same. We may conclude, then, that there is a real constant a such that $u(x, y) = a$ throughout D. Similarly, $v(x, y) = b$; and we find that $f(z) = a + bi$ at each point in D.

24. EXAMPLES

As pointed out in Sec. 23, it is often possible to determine where a given function is analytic by simply recalling various differentiation formulas in Sec. 19.

EXAMPLE 1. The quotient

$$f(z) = \frac{z^3 + 4}{(z^2 - 3)(z^2 + 1)}$$

is evidently analytic throughout the z plane except for the singular points $z = \pm\sqrt{3}$ and $z = \pm i$. The analyticity is due to the existence of familiar differentiation formulas, which need be applied only if the expression for $f'(z)$ is wanted.

When a function is given in terms of its component functions $u(x, y)$ and $v(x, y)$, its analyticity can be demonstrated by direct application of the Cauchy–Riemann equations.

EXAMPLE 2. When

$$f(z) = \cosh x \cos y + i \sinh x \sin y,$$

the component functions are

$$u(x, y) = \cosh x \cos y \quad \text{and} \quad v(x, y) = \sinh x \sin y.$$

Because

$$u_x = \sinh x \cos y = v_y \quad \text{and} \quad u_y = -\cosh x \sin y = -v_x$$

everywhere, it is clear from the theorem in Sec. 21 that f is entire.

Finally, we illustrate how the theorems in the last four sections, in particular the one in Sec. 23, can be used to obtain some important properties of analytic functions.

EXAMPLE 3. Suppose that a function

$$f(z) = u(x, y) + iv(x, y)$$

and its conjugate

$$\overline{f(z)} = u(x, y) - iv(x, y)$$

are *both* analytic in a given domain D. It is easy to show that $f(z)$ must be constant throughout D.

To do this, we write $\overline{f(z)}$ as

$$\overline{f(z)} = U(x, y) + iV(x, y),$$

where

$$U(x, y) = u(x, y) \quad \text{and} \quad V(x, y) = -v(x, y). \tag{1}$$

Because of the analyticity of $f(z)$, the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \tag{2}$$

hold in D, according to the theorem in Sec. 20. Also, the analyticity of $\overline{f(z)}$ in D tells us that

$$U_x = V_y, \quad U_y = -V_x.$$

In view of relations (1), these last two equations can be written

$$u_x = -v_y, \quad u_y = v_x. \tag{3}$$

By adding corresponding sides of the first of equations (2) and (3), we find that $u_x = 0$ in D. Similarly, subtraction involving corresponding sides of the second of equations (2) and (3) reveals that $v_x = 0$. According to expression (8) in Sec. 20, then,

$$f'(z) = u_x + iv_x = 0 + i0 = 0;$$

and it follows from the theorem in Sec. 23 that $f(z)$ is constant throughout D.

EXERCISES

1. Apply the theorem in Sec. 21 to verify that each of these functions is entire:

(*a*) $f(z) = 3x + y + i(3y - x)$; (*b*) $f(z) = \sin x \cosh y + i \cos x \sinh y$;

(*c*) $f(z) = e^{-y} \sin x - ie^{-y} \cos x$; (*d*) $f(z) = (z^2 - 2)e^{-x}e^{-iy}$.

2. With the aid of the theorem in Sec. 20, show that each of these functions is nowhere analytic:

(*a*) $f(z) = xy + iy$; (*b*) $f(z) = 2xy + i(x^2 - y^2)$; (*c*) $f(z) = e^y e^{ix}$.

3. State why a composition of two entire functions is entire. Also, state why any *linear combination* $c_1 f_1(z) + c_2 f_2(z)$ of two entire functions, where c_1 and c_2 are complex constants, is entire.

4. In each case, determine the singular points of the function and state why the function is analytic everywhere except at those points:

(*a*) $f(z) = \dfrac{2z+1}{z(z^2+1)}$; (*b*) $f(z) = \dfrac{z^3+i}{z^2-3z+2}$; (*c*) $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$.

Ans. (*a*) $z = 0, \pm i$; (*b*) $z = 1, 2$; (*c*) $z = -2, -1 \pm i$.

5. According to Exercise 4(*b*), Sec. 22, the function

$$g(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, -\pi < \theta < \pi)$$

is analytic in its domain of definition, with derivative

$$g'(z) = \frac{1}{2g(z)}.$$

Show that the composite function $G(z) = g(2z - 2 + i)$ is analytic in the half plane $x > 1$, with derivative

$$G'(z) = \frac{1}{g(2z-2+i)}.$$

Suggestion: Observe that $\text{Re}(2z - 2 + i) > 0$ when $x > 1$.

6. Use results in Sec. 22 to verify that the function

$$g(z) = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $g'(z) = 1/z$. Then show that the composite function $G(z) = g(z^2 + 1)$ is analytic in the quadrant $x > 0$, $y > 0$, with derivative

$$G'(z) = \frac{2z}{z^2+1}.$$

Suggestion: Observe that $\text{Im}(z^2 + 1) > 0$ when $x > 0$, $y > 0$.

7. Let a function $f(z)$ be analytic in a domain D. Prove that $f(z)$ must be constant throughout D if

(*a*) $f(z)$ is real-valued for all z in D; (*b*) $|f(z)|$ is constant throughout D.

Suggestion: Use the Cauchy–Riemann equations and the theorem in Sec. 23 to prove part (*a*). To prove part (*b*), observe that

$$\overline{f(z)} = \frac{c^2}{f(z)} \quad \text{if} \quad |f(z)| = c \ (c \neq 0);$$

then use the main result in Example 3, Sec. 24.

25. HARMONIC FUNCTIONS

A real-valued function H of two real variables x and y is said to be *harmonic* in a given domain of the xy plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$H_{xx}(x, y) + H_{yy}(x, y) = 0, \tag{1}$$

known as *Laplace's equation.*

Harmonic functions play an important role in applied mathematics. For example, the temperatures $T(x, y)$ in thin plates lying in the xy plane are often harmonic. A function $V(x, y)$ is harmonic when it denotes an electrostatic potential that varies only with x and y in the interior of a region of three-dimensional space that is free of charges.

EXAMPLE 1. It is easy to verify that the function $T(x, y) = e^{-y} \sin x$ is harmonic in any domain of the xy plane and, in particular, in the semi-infinite vertical strip $0 < x < \pi$, $y > 0$. It also assumes the values on the edges of the strip that are indicated in Fig. 31. More precisely, it satisfies all of the conditions

$$T_{xx}(x, y) + T_{yy}(x, y) = 0,$$
$$T(0, y) = 0, \quad T(\pi, y) = 0,$$
$$T(x, 0) = \sin x, \quad \lim_{y\to\infty} T(x, y) = 0,$$

which describe steady temperatures $T(x, y)$ in a thin homogeneous plate in the xy plane that has no heat sources or sinks and is insulated except for the stated conditions along the edges.

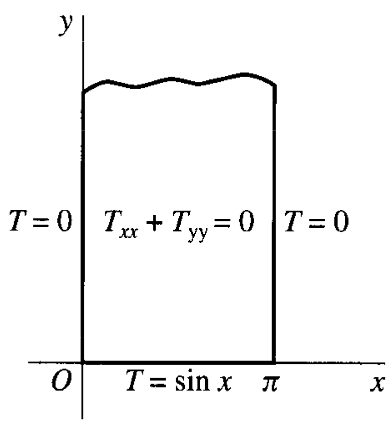

FIGURE 31

The use of the theory of functions of a complex variable in discovering solutions, such as the one in Example 1, of temperature and other problems is described in considerable detail later on in Chap. 10 and in parts of chapters following it.* That theory is based on the theorem below, which provides a source of harmonic functions.

Theorem 1. *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.*

To show this, we need a result that is to be proved in Chap. 4 (Sec. 48). Namely, if a function of a complex variable is analytic at a point, then its real and imaginary components have continuous partial derivatives of all orders at that point.

Assuming that f is analytic in D, we start with the observation that the first-order partial derivatives of its component functions must satisfy the Cauchy–Riemann equations throughout D:

$$u_x = v_y, \quad u_y = -v_x. \tag{2}$$

Differentiating both sides of these equations with respect to x, we have

$$u_{xx} = v_{yx}, \quad u_{yx} = -v_{xx}. \tag{3}$$

Likewise, differentiation with respect to y yields

$$u_{xy} = v_{yy}, \quad u_{yy} = -v_{xy}. \tag{4}$$

Now, by a theorem in advanced calculus,† the continuity of the partial derivatives of u and v ensures that $u_{yx} = u_{xy}$ and $v_{yx} = v_{xy}$. It then follows from equations (3) and (4) that

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0.$$

That is, u and v are harmonic in D.

EXAMPLE 2. The function $f(z) = e^{-y}\sin x - ie^{-y}\cos x$ is entire, as is shown in Exercise 1(*c*), Sec. 24. Hence its real part, which is the temperature function $T(x, y) = e^{-y}\sin x$ in Example 1, must be harmonic in every domain of the xy plane.

EXAMPLE 3. Since the function $f(z) = i/z^2$ is analytic whenever $z \neq 0$ and since

$$\frac{i}{z^2} = \frac{i}{z^2} \cdot \frac{\bar{z}^2}{\bar{z}^2} = \frac{i\bar{z}^2}{(z\bar{z})^2} = \frac{i\bar{z}^2}{|z|^4} = \frac{2xy + i(x^2 - y^2)}{(x^2 + y^2)^2},$$

* Another important method is developed in the authors' "Fourier Series and Boundary Value Problems," 6th ed., 2001.

† See, for instance, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 199–201, 1983.

the two functions

$$u(x, y) = \frac{2xy}{(x^2 + y^2)^2} \quad \text{and} \quad v(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

are harmonic throughout any domain in the xy plane that does not contain the origin.

If two given functions u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy–Riemann equations (2) throughout D, v is said to be a *harmonic conjugate* of u. The meaning of the word conjugate here is, of course, different from that in Sec. 5, where $\overline{z}$ is defined.

Theorem 2. *A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.*

The proof is easy. If v is a harmonic conjugate of u in D, the theorem in Sec. 21 tells us that f is analytic in D. Conversely, if f is analytic in D, we know from Theorem 1 above that u and v are harmonic in D; and, in view of the theorem in Sec. 20, the Cauchy–Riemann equations are satisfied.

The following example shows that if v is a harmonic conjugate of u in some domain, it is *not*, in general, true that u is a harmonic conjugate of v there. (See also Exercises 3 and 4.)

EXAMPLE 4. Suppose that

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Since these are the real and imaginary components, respectively, of the entire function $f(z) = z^2$, we know that v is a harmonic conjugate of u throughout the plane. But u cannot be a harmonic conjugate of v since, as verified in Exercise 2(*b*), Sec. 24, the function $2xy + i(x^2 - y^2)$ is not analytic anywhere.

In Chap. 9 (Sec. 97) we shall show that a function u which is harmonic in a domain of a certain type always has a harmonic conjugate. Thus, in such domains, every harmonic function is the real part of an analytic function. It is also true that a harmonic conjugate, when it exists, is unique except for an additive constant.

EXAMPLE 5. We now illustrate one method of obtaining a harmonic conjugate of a given harmonic function. The function

$$u(x, y) = y^3 - 3x^2y \tag{5}$$

is readily seen to be harmonic throughout the entire xy plane. Since a harmonic conjugate $v(x, y)$ is related to $u(x, y)$ by means of the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x, \tag{6}$$

the first of these equations tells us that

$$v_y(x, y) = -6xy.$$

Holding x fixed and integrating each side here with respect to y, we find that

$$v(x, y) = -3xy^2 + \phi(x), \tag{7}$$

where ϕ is, at present, an arbitrary function of x. Using the second of equations (6), we have

$$3y^2 - 3x^2 = 3y^2 - \phi'(x),$$

or $\phi'(x) = 3x^2$. Thus $\phi(x) = x^3 + C$, where C is an arbitrary real number. According to equation (7), then, the function

$$v(x, y) = -3xy^2 + x^3 + C \tag{8}$$

is a harmonic conjugate of $u(x, y)$.

The corresponding analytic function is

$$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + C). \tag{9}$$

The form $f(z) = i(z^3 + C)$ of this function is easily verified and is suggested by noting that when $y = 0$, expression (9) becomes $f(x) = i(x^3 + C)$.

EXERCISES

1. Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when

(*a*) $u(x, y) = 2x(1 - y)$; (*b*) $u(x, y) = 2x - x^3 + 3xy^2$;

(*c*) $u(x, y) = \sinh x \sin y$; (*d*) $u(x, y) = y/(x^2 + y^2)$.

Ans. (*a*) $v(x, y) = x^2 - y^2 + 2y$; (*b*) $v(x, y) = 2y - 3x^2y + y^3$;

(*c*) $v(x, y) = -\cosh x \cos y$; (*d*) $v(x, y) = x/(x^2 + y^2)$.

2. Show that if v and V are harmonic conjugates of u in a domain D, then $v(x, y)$ and $V(x, y)$ can differ at most by an additive constant.

3. Suppose that, in a domain D, a function v is a harmonic conjugate of u and also that u is a harmonic conjugate of v. Show how it follows that both $u(x, y)$ and $v(x, y)$ must be constant throughout D.

4. Use Theorem 2 in Sec. 25 to show that, in a domain D, v is a harmonic conjugate of u if and only if $-u$ is a harmonic conjugate of v. (Compare the result obtained in Exercise 3.)

Suggestion: Observe that the function $f(z) = u(x, y) + iv(x, y)$ is analytic in D if and only if $-if(z)$ is analytic there.

5. Let the function $f(z) = u(r, \theta) + iv(r, \theta)$ be analytic in a domain D that does not include the origin. Using the Cauchy–Riemann equations in polar coordinates (Sec. 22) and assuming continuity of partial derivatives, show that, throughout D, the function $u(r, \theta)$ satisfies the partial differential equation

$$r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0,$$

which is the *polar form of Laplace's equation.* Show that the same is true of the function $v(r, \theta)$.

6. Verify that the function $u(r, \theta) = \ln r$ is harmonic in the domain $r > 0, 0 < \theta < 2\pi$ by showing that it satisfies the polar form of Laplace's equation, obtained in Exercise 5. Then use the technique in Example 5, Sec. 25, but involving the Cauchy–Riemann equations in polar form (Sec. 22), to derive the harmonic conjugate $v(r, \theta) = \theta$. (Compare Exercise 6, Sec. 24.)

7. Let the function $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D, and consider the families of *level curves* $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if $z_0 = (x_0, y_0)$ is a point in D which is common to two particular curves $u(x, y) = c_1$ and $v(x, y) = c_2$ and if $f'(z_0) \neq 0$, then the lines tangent to those curves at (x_0, y_0) are perpendicular.

Suggestion: Note how it follows from the equations $u(x, y) = c_1$ and $v(x, y) = c_2$ that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0.$$

8. Show that when $f(z) = z^2$, the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ of the component functions are the hyperbolas indicated in Fig. 32. Note the orthogonality of the two

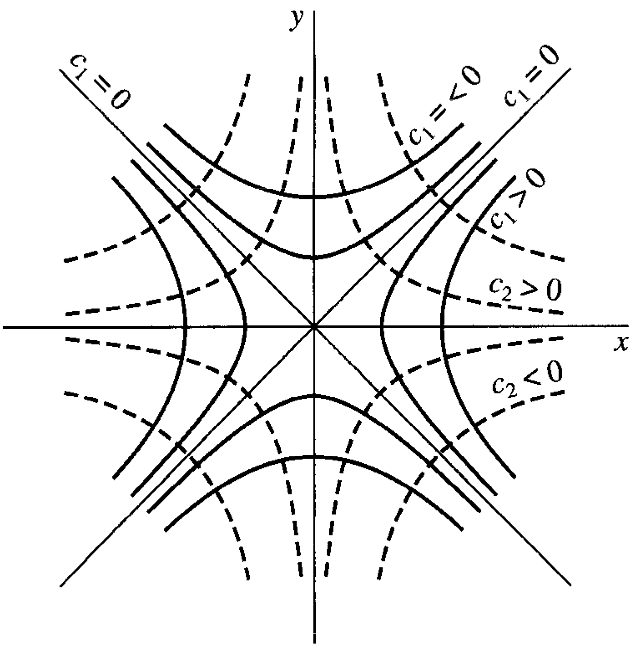

FIGURE 32

families, described in Exercise 7. Observe that the curves $u(x, y) = 0$ and $v(x, y) = 0$ intersect at the origin but are not, however, orthogonal to each other. Why is this fact in agreement with the result in Exercise 7?

9. Sketch the families of level curves of the component functions u and v when $f(z) = 1/z$, and note the orthogonality described in Exercise 7.

10. Do Exercise 9 using polar coordinates.

11. Sketch the families of level curves of the component functions u and v when

$$f(z) = \frac{z-1}{z+1},$$

and note how the result in Exercise 7 is illustrated here.

26. UNIQUELY DETERMINED ANALYTIC FUNCTIONS

We conclude this chapter with two sections dealing with how the values of an analytic function in a domain D are affected by its values in a subdomain or on a line segment lying in D. While these sections are of considerable theoretical interest, they are not central to our development of analytic functions in later chapters. The reader may pass directly to Chap. 3 at this time and refer back when necessary.

Lemma. *Suppose that*
(i) a function f is analytic throughout a domain D;
(ii) $f(z) = 0$ at each point z of a domain or line segment contained in D.
Then $f(z) \equiv 0$ in D; that is, $f(z)$ is identically equal to zero throughout D.

To prove this lemma, we let f be as stated in its hypothesis and let z_0 be any point of the subdomain or line segment at each point of which $f(z) = 0$. Since D is a *connected* open set (Sec. 10), there is a polygonal line L, consisting of a finite number of line segments joined end to end and lying entirely in D, that extends from z_0 to any other point P in D. We let d be the shortest distance from points on L to the boundary of D, unless D is the entire plane; in that case, d may be any positive number. We then form a finite sequence of points

$$z_0, z_1, z_2, \ldots, z_{n-1}, z_n$$

along L, where the point z_n coincides with P (Fig. 33) and where each point is sufficiently close to the adjacent ones that

$$|z_k - z_{k-1}| < d \qquad (k = 1, 2, \ldots, n).$$

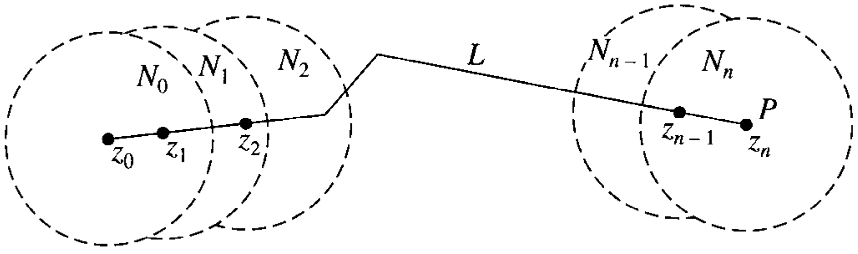

FIGURE 33

Finally, we construct a finite sequence of neighborhoods

$$N_0, N_1, N_2, \ldots, N_{n-1}, N_n,$$

where each neighborhood N_k is centered at z_k and has radius d. Note that these neighborhoods are all contained in D and that the center z_k of any neighborhood N_k ($k = 1, 2, \ldots, n$) lies in the preceding neighborhood N_{k-1}.

At this point, we need to use a result that is proved later on in Chap. 6. Namely, Theorem 3 in Sec. 68 tells us that since f is analytic in the domain N_0 and since $f(z) = 0$ in a domain or on a line segment containing z_0, then $f(z) \equiv 0$ in N_0. But the point z_1 lies in the domain N_0. Hence a second application of the same theorem reveals that $f(z) \equiv 0$ in N_1; and, by continuing in this manner, we arrive at the fact that $f(z) \equiv 0$ in N_n. Since N_n is centered at the point P and since P was arbitrarily selected in D, we may conclude that $f(z) \equiv 0$ in D. This completes the proof of the lemma.

Suppose now that two functions f and g are analytic in the same domain D and that $f(z) = g(z)$ at each point z of some domain or line segment contained in D. The difference

$$h(z) = f(z) - g(z)$$

is also analytic in D, and $h(z) = 0$ throughout the subdomain or along the line segment. According to the above lemma, then, $h(z) = 0$ throughout D; that is, $f(z) = g(z)$ at each point z in D. We thus arrive at the following important theorem.

Theorem. *A function that is analytic in a domain D is uniquely determined over D by its values in a domain, or along a line segment, contained in D.*

This theorem is useful in studying the question of extending the domain of definition of an analytic function. More precisely, given two domains D_1 and D_2, consider the *intersection* $D_1 \cap D_2$, consisting of all points that lie in both D_1 and D_2. If D_1 and D_2 have points in common (see Fig. 34) and a function f_1 is analytic in D_1, there *may* exist a function f_2, which is analytic in D_2, such that $f_2(z) = f_1(z)$ for each z in the intersection $D_1 \cap D_2$. If so, we call f_2 an *analytic continuation* of f_1 into the second domain D_2.

Whenever that analytic continuation exists, it is unique, according to the theorem just proved. That is, not more than one function can be analytic in D_2 and assume the

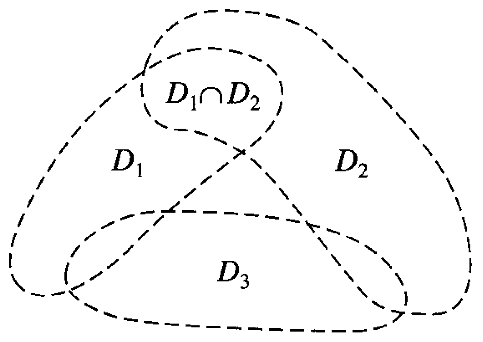

FIGURE 34

value $f_1(z)$ at each point z of the domain $D_1 \cap D_2$ interior to D_2. However, if there is an analytic continuation f_3 of f_2 from D_2 into a domain D_3 which intersects D_1, as indicated in Fig. 34, it is not necessarily true that $f_3(z) = f_1(z)$ for each z in $D_1 \cap D_3$. Exercise 2, Sec. 27, illustrates this.

If f_2 is the analytic continuation of f_1 from a domain D_1 into a domain D_2, then the function F defined by the equations

$$F(z) = \begin{cases} f_1(z) & \text{when } z \text{ is in } D_1, \\ f_2(z) & \text{when } z \text{ is in } D_2 \end{cases}$$

is analytic in the *union* $D_1 \cup D_2$, which is the domain consisting of all points that lie in either D_1 or D_2. The function F is the analytic continuation into $D_1 \cup D_2$ of either f_1 or f_2; and f_1 and f_2 are called *elements* of F.

27. REFLECTION PRINCIPLE

The theorem in this section concerns the fact that some analytic functions possess the property that $\overline{f(z)} = f(\overline{z})$ for all points z in certain domains, while others do not. We note, for example, that $z + 1$ and z^2 have that property when D is the entire finite plane; but the same is not true of $z + i$ and iz^2. The theorem, which is known as the *reflection principle,* provides a way of predicting when $\overline{f(z)} = f(\overline{z})$.

Theorem. *Suppose that a function f is analytic in some domain D which contains a segment of the x axis and whose lower half is the reflection of the upper half with respect to that axis. Then*

$$\overline{f(z)} = f(\overline{z}) \tag{1}$$

for each point z in the domain if and only if $f(x)$ is real for each point x on the segment.

We start the proof by assuming that $f(x)$ is real at each point x on the segment. Once we show that the function

$$F(z) = \overline{f(\overline{z})} \tag{2}$$

is analytic in D, we shall use it to obtain equation (1). To establish the analyticity of $F(z)$, we write

$$f(z) = u(x, y) + iv(x, y), \quad F(z) = U(x, y) + iV(x, y)$$

and observe how it follows from equation (2) that, since

$$\overline{f(\overline{z})} = u(x, -y) - iv(x, -y), \tag{3}$$

the components of $F(z)$ and $f(z)$ are related by the equations

$$U(x, y) = u(x, t) \quad \text{and} \quad V(x, y) = -v(x, t), \tag{4}$$

where $t = -y$. Now, because $f(x + it)$ is an analytic function of $x + it$, the first-order partial derivatives of the functions $u(x, t)$ and $v(x, t)$ are continuous throughout D and satisfy the Cauchy–Riemann equations*

$$u_x = v_t, \quad u_t = -v_x. \tag{5}$$

Furthermore, in view of equations (4),

$$U_x = u_x, \quad V_y = -v_t \frac{dt}{dy} = v_t;$$

and it follows from these and the first of equations (5) that $U_x = V_y$. Similarly,

$$U_y = u_t \frac{dt}{dy} = -u_t, \quad V_x = -v_x;$$

and the second of equations (5) tells us that $U_y = -V_x$. Inasmuch as the first-order partial derivatives of $U(x, y)$ and $V(x, y)$ are now shown to satisfy the Cauchy–Riemann equations and since those derivatives are continuous, we find that the function $F(z)$ is analytic in D. Moreover, since $f(x)$ is real on the segment of the real axis lying in D, $v(x, 0) = 0$ on that segment; and, in view of equations (4), this means that

$$F(x) = U(x, 0) + iV(x, 0) = u(x, 0) - iv(x, 0) = u(x, 0).$$

That is,

$$F(z) = f(z) \tag{6}$$

at each point on the segment. We now refer to the theorem in Sec. 26, which tells us that an analytic function defined on a domain D is uniquely determined by its values along any line segment lying in D. Thus equation (6) actually holds throughout D.

* See the paragraph immediately following Theorem 1 in Sec. 25.

Because of definition (2) of the function $F(z)$, then,

(7) $$\overline{f(\overline{z})} = f(z);$$

and this is the same as equation (1).

To prove the converse of the theorem, we assume that equation (1) holds and note that, in view of expression (3), the form (7) of equation (1) can be written

$$u(x, -y) - iv(x, -y) = u(x, y) + iv(x, y).$$

In particular, if $(x, 0)$ is a point on the segment of the real axis that lies in D,

$$u(x, 0) - iv(x, 0) = u(x, 0) + iv(x, 0);$$

and, by equating imaginary parts here, we see that $v(x, 0) = 0$. Hence $f(x)$ is real on the segment of the real axis lying in D.

EXAMPLES. Just prior to the statement of the theorem, we noted that

$$\overline{z+1} = \overline{z} + 1 \quad \text{and} \quad \overline{z^2} = \overline{z}^2$$

for all z in the finite plane. The theorem tells us, of course, that this is true, since $x + 1$ and x^2 are real when x is real. We also noted that $z + i$ and iz^2 do not have the reflection property throughout the plane, and we now know that this is because $x + i$ and ix^2 are *not* real when x is real.

EXERCISES

1. Use the theorem in Sec. 26 to show that if $f(z)$ is analytic and not constant throughout a domain D, then it cannot be constant throughout any neighborhood lying in D.

Suggestion: Suppose that $f(z)$ does have a constant value w_0 throughout some neighborhood in D.

2. Starting with the function

$$f_1(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, 0 < \theta < \pi)$$

and referring to Exercise 4(*b*), Sec. 22, point out why

$$f_2(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, \frac{\pi}{2} < \theta < 2\pi\right)$$

is an analytic continuation of f_1 across the negative real axis into the lower half plane. Then show that the function

$$f_3(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0, \pi < \theta < \frac{5\pi}{2}\right)$$

is an analytic continuation of f_2 across the positive real axis into the first quadrant but that $f_3(z) = -f_1(z)$ there.

3. State why the function

$$f_4(z) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, -\pi < \theta < \pi)$$

is the analytic continuation of the function $f_1(z)$ in Exercise 2 across the positive real axis into the lower half plane.

4. We know from Example 1, Sec. 21, that the function

$$f(z) = e^x e^{iy}$$

has a derivative everywhere in the finite plane. Point out how it follows from the reflection principle (Sec. 27) that

$$\overline{f(z)} = f(\overline{z})$$

for each z. Then verify this directly.

5. Show that if the condition that $f(x)$ is real in the reflection principle (Sec. 27) is replaced by the condition that $f(x)$ is pure imaginary, then equation (1) in the statement of the principle is changed to

$$\overline{f(z)} = -f(\overline{z}).$$